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Linear operator pencils on Lie algebras and Laurent biorthogonal polynomials

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Abstract

We study operator pencils on generators of the Lie algebras sl_2 and the oscillator algebra. These pencils are linear in a spectral parameter λ. The corresponding generalized eigenvalue problem gives rise to some sets of orthogonal polynomials and Laurent biorthogonal polynomials (LBP) expressed in terms of the Gauss ${}_2F_1$ and degenerate ${}_1F_1$ hypergeometric functions. For special choices of the parameters of the pencils, we identify the resulting polynomials with the Hendriksen–van Rossum LBP which are widely believed to be the biorthogonal analogues of the classical orthogonal polynomials. This places these examples under the umbrella of the *generalized bispectral problem* which is considered here. Other (non-bispectral) cases give rise to some 'nonclassical' orthogonal polynomials including Tricomi–Carlitz and random-walk polynomials. An application to solutions of relativistic Toda chain is considered.

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1. Introduction

The so-called *bispectral problem* consists of finding situations where a function $\psi(x, z)$ satisfies

$$L\psi = z\psi \tag{1.1}$$

and

$$B\psi = \theta(x)\psi. \tag{1.2}$$

Here L and B are differential or difference operators in the variables x and z respectively, and $\theta(x)$ is a function. In this generality the problem has surfaced in many different places. We mention here (a) the work of Bochner [8] pursued later by Krall [17], where x takes on the values $n = 0, 1, 2, \ldots$ and $\psi(n, z)$ is assumed to be a polynomial in z of degree n, (b) the work of Duistermaat and Grünbaum [9] where both variables are continuous and L has order two, (c) the work of Wilson [30] as well as that of Bakalov *et al* [5] where this restriction on the order of L is lifted, (d) the work of Grünbaum and Haine [13] where the Bochner problem is revisited removing the polynomial restriction on $\psi(n, z)$, (e) the work of Leonard [18] where both variables are discrete and the matrices L and B are tridiagonal and finally (f) the following papers by Spiridonov *et al* [22–24]. A natural generalization of this problem, along the lines in [32, 25] is to ask for all situations where

$$L\psi = zM\psi \tag{1.3}$$

and

$$B\psi = \theta(x)C\psi. \tag{1.4}$$

Here L and M are local operators in the variable x and B as well as C are local operators in the variable z. The spectral problem of the type (1.3) is called the *generalized eigenvalue problem* (GEVP) in the mathematical literature [29]. Alternatively this problem is called a diagonalization of a linear operator pencil, because (1.3) can be rewritten in the form $W(z)\psi = 0$, where $W(z) = L - zM$ is a *linear pencil*, i.e. a linear combination of two operators L, M with spectral parameter z. For the case of matrices (i.e. L, M are finite-dimensional operators) the theory of linear pencils can be found e.g. in the classical monograph [11].

It is well known that for tri-diagonal matrices L the ordinary eigenvalue problem (1.1) is equivalent to the theory of orthogonal polynomials [7]. It was shown in [32] that for two tri-diagonal matrices L, M the GEVP (1.3) is equivalent to the theory of biorthogonal rational functions (BRF). From the Leonard theorem [18] it is known that all bispectral pairs (1.1), (1.2) where both operators L and B are finite-dimensional matrices are reduced to the theory of the q-Racah polynomials and their specializations (such as Hahn, Krawtchouk etc polynomials). For the case of generalized bispectral pair (1.3), (1.4), when all operators L, M, B, C are finite-dimensional matrices, the problem is still open. Nevertheless it is believed that the so-called elliptic BRF proposed in [25] are the most general objects appearing in such bispectral problems.

The generalized eigenvalue problems of type (1.3) and (1.4) are important in some problems in mechanics [11, 29]. It was also recognized that GEVP can be used in order to describe Lax pair representation for some integrable chains like relativistic Toda chain (RTC) and its generalizations [27].

A natural generalization of the linear pencil (1.3) is the so-called *multiparameter pencil* [4]:

$$\left(\sum_{i=0}^{N} \lambda_i L_i\right) \psi = 0 \tag{1.5}$$

where λ_i are scalars and L_i are operators. The linear pencil (1.3) is obtained if $N = 1$. Note that multiparameter pencils appear, e.g., in separation of variables in the theory of partial differential equations [2].

A special case of the multiparameter pencil is the *polynomial pencil* when $\lambda_k = \lambda^k, k = 0, 1, \ldots, N$. The polynomial eigenvalue problem arises in the theory of the vibrations of mechanical systems under external forces, in the simulation of electronic circuits, and in fluid mechanics [15].

For the case of finite-dimensional operators, the polynomial pencil is known under the name λ-*matrices* [11], i.e. the set of matrices $A(\lambda)$ being polynomials in a parameter λ. Note that λ-matrices are important tool in the theory of canonical reduction of matrices to Jordan form [11].

In this paper we will see how the study of representations of sl_2 leads to some instances of the situation described above. This means that the operators L_i in the polynomial pencil belong to sl_2 representations. For the case of the ordinary eigenvalue problem (1.1) it is well known [12, 10] that the corresponding eigenfunctions $\psi(z)$ can be expressed in terms of the orthogonal polynomials of the Sheffer type, i.e. Meixner, Pollaczeck and Laguerre polynomials for the sl_2 case and Hermite and Charlier polynomials for the case of the oscillator algebra. However, as far as we know, the case of linear (and polynomial) pencil for these algebras has not yet been investigated in the literature.

The paper is organized as follows. In section 2 we recall well-known results concerning representations of sl_2 and the oscillator algebra. In section 3 we consider a general polynomial ansatz for the sl_2 algebra and find explicit expression for corresponding eigenfunctions in terms of the Gauss hypergeometric function ${}_2F_1$. In section 4 we find a condition under which our polynomial pencil is bispectral in a special sense, i.e. the expansion coefficients $C_n(\lambda)$ satisfy a linear *differential equation* with respect to λ. Section 5 is devoted to an important special case of the linear pencil. This means that the problem can be reduced to the so-called generalized eigenvalue problem $L\psi = \lambda M\psi$ for two operators L, M belonging to a representation of the sl_2 algebra. In generic case the coefficients $C_n(\lambda)$ are expressed in terms of the so-called Laurent biorthogonal polynomials (LBP). We find that the bispectrality condition leads to the 'classical' LBP found by Hendriksen and van Rossum [14]. In section 6 we consider a special case of the linear pencil on sl_2 when the coefficients $C_n(\lambda)$ are expressed as orthogonal polynomials in λ. We recover well-known orthogonal polynomials of the Sheffer type (i.e. Meixner, Pollaczeck, Laguerre, Charlier and Hermite) and some new orthogonal polynomials. In the symmetric case these OP are reduced to the so-called 'random-walk' polynomials studied in [3]. In section 7 we describe briefly the situation for the oscillator algebra where we see that some new cases of 'non-classical' LBP and OP arise. In section 8 we give an interesting application of our results to integrable chains such as relativistic Toda chain.

2. Representations of the sl_2 and oscillator algebra

Let K_0, K_-, K_+ be generators of the sl_2 Lie algebra satisfying the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0. \tag{2.1}$$

There are two real forms of the sl_2 algebra: the Lie algebras $su(1, 1)$ and $su(2)$. The unitary representations of the algebra $su(1, 1)$ correspond to the case when the operator K_0 is Hermitian in some Hilbert space whereas the operators K_-, K_+ are conjugated: $K_+ = K_-^\dagger$. The unitary representations of the $su(2)$ algebra correspond to the case when the operator K_0 is Hermitian and $K_+ = -K_-^\dagger$. For the $su(2)$ case it is convenient to introduce slightly modified generators $J_0 = K_0$, $J_- = K_-$, $J_+ = -K_+$. Then we have the commutation relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_-, J_+] = -2J_0 \tag{2.2}$$

which differ from the relations (2.1) only by the change of the sign in the last relation.

The sl_2 algebra has the Casimir element

$$Q = K_0^2 - K_0 - K_+K_- \tag{2.3}$$

commuting with all generators $[Q, K_0] = [Q, K_\pm] = 0$.

For the irreducible representations of the $su(1,1)$ algebra belonging to the positive discrete series we have $Q = \nu(\nu-1)$, where ν is a real positive parameter, $\nu > 0$. There exists a basis $e_n, n = 0, 1, \ldots$ such that operators $K_0, K_\pm$ act as follows

$$K_0 e_n = (n+\nu)e_n \qquad K_- e_n = (\rho_n)^{1/2} e_{n-1} \qquad K_+ e_n = (\rho_{n+1})^{1/2} e_{n+1} \tag{2.4}$$

where $\rho_n = n(n+2\nu-1)$. It is clear that under condition $\nu > 0$ one has $\rho_n > 0, n = 1, 2, \ldots$ and hence the representation (2.4) indeed defines the Hermitian operator K_0 and Hermitian conjugated operators K_- and K_+. This representation is infinite dimensional: $n = 0, 1, \ldots$. The vector e_0 is the so-called vacuum vector because it is annihilated by the operator K_-.

For the irreducible representations of the $su(2)$ algebra we always have a finite-dimensional representation

$$J_0 e_n = (n-j)e_n \qquad J_- e_n = (\rho_n)^{1/2} e_{n-1} \qquad J_+ e_n = (\rho_{n+1})^{1/2} e_{n+1} \tag{2.5}$$

where $\rho_n = n(2j - n + 1)$. The parameter j can take only integer and half-integer positive values: $j = 1/2, 1, 3/2, 2, \ldots$. The integer $2j+1$ is the dimension of the representation (2.5).

The oscillator algebra can be considered as a contraction of the sl_2 algebra under the limiting process $\nu \to \infty$. It consists of four generators N_0, N_-, N_+ and the neutral element (unity) satisfying the commutation relations

$$[N_0, N_\pm] = \pm N_\pm \qquad [N_-, N_+] = 1. \tag{2.6}$$

Irreducible representations of the oscillator algebra are defined as

$$N_0 e_n = (n+c)e_n \qquad N_- e_n = \sqrt{n} e_{n-1} \qquad N_+ e_n = \sqrt{n+1} e_{n+1} \tag{2.7}$$

where c is an arbitrary real parameter. In fact, it is always possible to put $c = 0$ without loss of generality. In this case there is a natural equivalence between the oscillator algebra and the Heisenberg–Weyl algebra, defined by three generators $a, a^\dagger, 1$ with commutation relations

$$[a, a^\dagger] = 1 \qquad [a, 1] = [a^\dagger, 1] = 0. \tag{2.8}$$

This equivalence is provided by the formulae: $N_0 = a^\dagger a$, $N_- = a$, $N_+ = a^\dagger$.

3. Polynomial pencil on sl_2

Consider the problem

$$K(\lambda)\psi(\lambda) = 0 \tag{3.1}$$

where the operator $K(\lambda)$

$$K(\lambda) = A_0(\lambda)K_0 + A_1(\lambda)K_- + A_2(\lambda)K_+ + A_3(\lambda) \tag{3.2}$$

is a linear combination of the generators K_0, K_-, K_+ of the sl_2 algebra with the coefficients $A_i(\lambda), i = 0, 1, 2, 3$ being polynomials in some parameter λ.

The vector $\psi(\lambda)$ can be expanded in terms of the basis e_n:

$$\psi(\lambda) = \sum_{k=0}^{\infty} C_k(\lambda) e_k. \tag{3.3}$$

For $C_k(\lambda)$ we obtain a recurrence relation from (3.1)

$$A_1(\lambda)\sqrt{(n+1)(n+2\nu)}C_{n+1}(\lambda) + A_2(\lambda)\sqrt{n(n+2\nu-1)}C_{n-1}(\lambda) + (A_0(\lambda)(n+\nu) + A_3(\lambda))C_n(\lambda) = 0 \quad n = 0, 1, \ldots. \tag{3.4}$$

From (3.4) it is clear that all coefficients $C_k(\lambda)$, $k = 1, 2, \ldots$ are expressed uniquely through only one (arbitrary) coefficient $C_0(\lambda)$. If $C_0(\lambda) = 1$ (standard choice) then from (3.4) it follows that all the expansion coefficients $C_n(\lambda)$ are rational functions in the spectral parameter λ.

It is convenient to renormalize the coefficients

$$C_n(\lambda) = \sqrt{\frac{n!}{(2\nu)_n}} E_n(\lambda). \tag{3.5}$$

Then we get a recurrence relation for the coefficients $E_n(z)$

$$(n+1)A_1(\lambda)E_{n+1}(\lambda) + ((n+\nu)A_0(\lambda) + A_3(\lambda))E_n(\lambda) + (n+2\nu-1)A_2(\lambda)E_{n-1}(\lambda) = 0. \tag{3.6}$$

Define now the generating function

$$F(z;\lambda) = \sum_{n=0}^{\infty} E_n(\lambda) z^n \tag{3.7}$$

for the coefficients $E_n(\lambda)$. Assume that the second degree polynomial $A_2z^2 + A_0z + A_1$ has distinct roots: $A_2z^2 + A_0z + A_1 = A_2(z - z_1)(z - z_2)$.

It is easily seen that the function $F(z)$ satisfies a first-order differential equation with respect to z:

$$\frac{F_z}{F} = -\frac{2A_2(\lambda)\nu z + A_3(\lambda) + \nu A_0(\lambda)}{A_2(\lambda)z^2 + A_0(\lambda)z + A_1(\lambda)} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} \tag{3.8}$$

where the parameters α_1, α_2 are defined by the relations

$$\alpha_1 + \alpha_2 = -2\nu \qquad A_2(\alpha_1 z_2 + \alpha_2 z_1) = A_3 + \nu A_0. \tag{3.9}$$

Hence, we have for $F(z)$ a simple expression

$$F(z) = E_0(\lambda)(1 - z/z_1)^{\alpha_1}(1 - z/z_2)^{\alpha_2}. \tag{3.10}$$

Expanding the binomials in (3.10) in terms of z we arrive at the expression

$$F(z) = E_0(\lambda) \sum_{n,m=0}^{\infty} \frac{(-\alpha_1)_n(-\alpha_2)_m z^{n+m}}{n!m!z_1^n z_2^m}. \tag{3.11}$$

Rearranging terms in (3.11) we finally get

$$E_n(\lambda) = E_0(\lambda)\frac{(-\alpha_2)_n}{n!z_2^n} {}_2F_1\left(\begin{matrix} -n, -\alpha_1 \\ 1 + \alpha_2 - n \end{matrix}; z_2/z_1\right). \tag{3.12}$$

Since $C_0(\lambda) = E_0(\lambda)$ we can put $C_0 = E_0 = 1$ without loss of generality. Then for $C_n(\lambda)$ we obtain from (3.5)

$$C_n(\lambda) = \sqrt{\frac{n!}{(2\nu)_n}} \frac{(-\alpha_2)_n}{n!z_2^n} {}_2F_1\left(\begin{matrix} -n, -\alpha_1 \\ 1 + \alpha_2 - n \end{matrix}; z_2/z_1\right). \tag{3.13}$$

Thus we get that in the general case (when the roots z_1, z_2 are distinct), the coefficients $C_n(\lambda)$ are expressed in terms of the Gauss hypergeometric function ${}_2F_1$.

We consider now the degenerate case when the roots of the polynomial $A_2z^2 + A_0z + A_1$ coincide, i.e. $A_2z^2 + A_0z + A_1 = A_2(z - z_1)^2$. In this case we have the differential equation

$$\frac{F_z}{F} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{(z - z_1)^2} \tag{3.14}$$

where

$$\alpha_1 = -2\nu \qquad \alpha_2 = -A_3/A_2.$$

The solution of (3.14) (with initial condition $F(0) = 1$) is

$$F(z) = (1-y)^{-2\nu}\exp(\gamma y/(1-y)) \tag{3.15}$$

where $y = z/z_1$ and $\gamma = -A_3/(A_2 z_1) = 2A_3/A_0$. Expanding the function $F(y)$ in terms of y we easily obtain in this case

$$C_n(\lambda) = \frac{(2\nu)_n(-2A_2/A_0)^n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\2\nu\end{matrix};-2A_3/A_0\right). \tag{3.16}$$

4. Bispectrality. Dual polynomial pencils

We see that the coefficients $C_n(\lambda)$ on the one hand satisfy the recurrence relation (3.4) corresponding to a polynomial pencil on sl_2. On the other hand, these coefficients are expressed in terms of hypergeometric functions which are known to satisfy differential equations with respect to their argument. It is interesting, therefore to consider the cases when coefficients $C_n(\lambda)$ are bispectral, i.e. they satisfy conditions of polynomial pencil of type

$$B_2(v;\mu_n)\frac{\mathrm{d}^2 C_n(\lambda(v))}{\mathrm{d}v^2} + B_1(v,\mu_n)\frac{\mathrm{d}C_n(\lambda(v))}{\mathrm{d}v} + B_0(v,\mu_n)C_n(\lambda(v)) = 0 \tag{4.1}$$

where v is a new variable and $B_i(v,n)$ are functions of two variables v, n with the condition that they are polynomials in some variable (generalized eigenvalue) μ_n. In this case we have a polynomial pencil and one can say about bispectrality: the coefficients $C_n(\lambda)$ satisfy conditions of polynomial pencil with respect to two dual variables. Special cases of linear pencils of type (4.1) were considered by Sawyer [21] and Chaundy [6].

Consider first the nondegenerate case described by formula (3.13). It is clear that $C_n(\lambda)$ is a rational function in λ. On the other hand, the hypergeometric function

$$F(y) = {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};y\right)$$

with parameters independent of y, satisfies the second-order differential equation

$$y(1-y)F''(y) + (c-(a+b+1)y)F'(y) - abF(y) = 0. \tag{4.2}$$

Hence, if the parameters α_1, α_2 do not depend on λ in (3.13), then the functions $C_n(\lambda)$ will satisfy differential equation of type (4.1).

Eliminating z_1 and z_2 from relations (3.9), we arrive at

Proposition 1. *The parameters α_1, α_2 do not depend on λ if and only if the polynomials $A_i(\lambda)$ satisfy the condition*

$$A_0(\lambda)^2 - 4A_1(\lambda)A_2(\lambda) = \kappa^2 A_3(\lambda)^2 \tag{4.3}$$

where κ is an arbitrary constant independent on λ.

Note that if condition (4.3) is fulfilled, we have

$$z_{1,2} = \frac{A_0(\lambda)\pm\kappa A_3(\lambda)}{2A_2(\lambda)}$$

and hence the argument z_2/z_1 of the hypergeometric function in (3.13) is a rational function of λ.

For the degenerate case (3.16) we see that parameters of the degenerated hypergeometric function ${}_1F_1$ do not depend on λ and hence, as is easily verified, the corresponding functions $C_n(\lambda)$ satisfy (4.1) with *quadratic dependence* in n if $A_2(\lambda)/A_0(\lambda) \neq \text{const}$ and a *linear* dependence in n if $A_2(\lambda)/A_0(\lambda) = \text{const}$.

5. Linear pencil and Laurent biorthogonal polynomials

Consider the special case of a *linear* pencil, i.e. we assume that all coefficients $A_i(\lambda)$ are linear functions in λ

$$A_i(\lambda) = a_i^{(0)} - \lambda a_i^{(1)} \qquad i = 0, 1, 2, 3. \tag{5.1}$$

In this case we have a typical generalized eigenvalue problem

$$L\psi(\lambda) = \lambda M\psi(\lambda) \tag{5.2}$$

where $M = a_1^{(0)}K_- + a_2^{(0)}K_+ + a_0^{(0)}K_0 + a_3^{(0)}$ and $M = a_1^{(1)}K_- + a_2^{(1)}K_+ + a_0^{(1)}K_0 + a_3^{(1)}$.

As was shown in [32] a GEVP of the type (5.2) generates two families of biorthogonal rational functions. However, in our special case connected with the sl_2 algebra these rational functions can be reduced to the so-called Laurent biorthogonal polynomials.

Indeed, the generalized eigenvalue problem (5.2) admits an obvious linear transformation of the initial operators [32]

$$\tilde{L}\psi(\lambda) = \tilde{\lambda}\tilde{M}\psi(\lambda) \tag{5.3}$$

where $\tilde{L} = \alpha L + \beta M, \tilde{M} = \gamma L + \delta M$ and $\tilde{\lambda} = \alpha\lambda + \beta/\gamma\lambda + \delta$ with arbitrary complex parameters $\alpha, \ldots, \delta$ such that $\alpha\delta - \beta\gamma \neq 0$. Using this observation we can choose linear transformations to reduce the GEVP (5.3) to the simplest form.

Assume first that

$$a_2^{(0)}a_1^{(1)} - a_1^{(0)}a_2^{(1)} \neq 0. \tag{5.4}$$

In particular, this condition means that we have nondegenerate case (3.13). Then it is possible to find unique coefficients $\alpha, \ldots, \delta$ such that

$$\tilde{a}_1^{(0)} = -\tilde{a}_2^{(1)} = 1 \qquad \tilde{a}_2^{(0)} = \tilde{a}_1^{(1)} = 0. \tag{5.5}$$

In what follows we will assume that conditions (5.5) hold. This means that the GEVP is reduced to the form

$$\left(K_- + a_0^{(0)}K_0 + a_3^{(0)}\right)\psi(\lambda) = \lambda\left(-K_+ + a_0^{(0)}K_0 + a_3^{(0)}\right)\psi(\lambda). \tag{5.6}$$

Condition (4.3) in this case is equivalent to

$$a_0^{(0)} = \mu \qquad a_3^{(1)} = \kappa\mu \qquad a_0^{(1)} = -1/\mu \qquad a_3^{(1)} = \kappa/\mu \tag{5.7}$$

where μ, κ are arbitrary nonzero constants. Hence in this case we can rewrite the GEVP in the form

$$(K_- + \mu K_0 + \kappa\mu)\psi(\lambda) = \lambda(-K_+ - \mu^{-1}K_0 + \kappa\mu^{-1})\psi(\lambda). \tag{5.8}$$

From (3.6) we get a recurrence relation for the coefficients $E_n(\lambda)$:

$$(n+1)E_{n+1} + (\mu(n+\nu) + \kappa\mu)E_n = \lambda(-(n+2\nu-1)E_{n-1} + \mu^{-1}(\kappa - n - \nu)E_n). \tag{5.9}$$

Introduce functions $P_n(\lambda)$ by

$$P_n(\lambda) = (-1)^n\frac{(\nu-\kappa)_n}{\mu^n}E_n(\lambda).$$

Then it is easily verified that $P_n(\lambda)$ are monic polynomials of degree n satisfying the recurrence relation

$$P_{n+1} + d_nP_n = \lambda(P_n + g_nP_{n-1}) \tag{5.10}$$

where

$$d_n = -\mu^2\frac{n+\nu+\kappa}{n+\nu-\kappa} \qquad g_n = -\mu^2\frac{n(n+2\nu-1)}{(n+\nu-\kappa)(n+\nu-\kappa-1)}. \tag{5.11}$$

The initial conditions are

$$P_0 = 1 \qquad P_1 = \lambda - d_0. \tag{5.12}$$

It is interesting to note that for generic non-zero coefficients d_n, g_n the recurrence relation (5.10) with initial conditions (5.12) determines the so-called monic Laurent biorthogonal polynomials $P_n(\lambda)$ [14]. In contrast to the case of orthogonal polynomials, LBP have the *biorthogonality* property [14]

$$\int_\Gamma P_n(z) P_m^*(1/z) w(z)\, \mathrm{d}z = h_n \delta_{mn} \tag{5.13}$$

with some function $w(z)$ and some contour Γ on complex plane. Here $P_n^*(z)$ are the so-called dual LBP which have the simple expression

$$P_n^*(z) = \frac{z^{-n} P_{n+1}(z) - z^{1-n} P_n(z)}{P_{n+1}(0)}. \tag{5.14}$$

The dual LBP satisfy a recurrence relation of the same type (5.10):

$$P_{n+1}^* + d_n^* P_n^* = \lambda(P_n^* + g_n^* P_{n-1}^*) \tag{5.15}$$

with recurrence coefficients [28]

$$d_n^* = \frac{d_{n+1} - g_{n+1}}{d_{n+1}(d_n - g_n)} \qquad g_n^* = \frac{g_n(d_{n+1} - g_{n+1})}{d_{n+1} d_n (d_n - g_n)}. \tag{5.16}$$

Simple calculation shows that dual recurrence coefficients d_n^*, g_n^* have the same expression (5.11) but with $\mu^* = 1/\mu, \kappa^* = -\kappa - 1$. Thus dual polynomials $P_n^*(\lambda)$ can be as well obtained from the linear pencil on sl_2 and moreover, the polynomials $P_n^*(\lambda)$ again satisfy the basic property (4.3) which guarantees that both $P_n(\lambda)$ and $P_n^*(\lambda)$ are bispectral, i.e. satisfy second-order differential equations in λ.

The LBP possess an obvious scaling transformation [28]. Assume that the polynomials $P_n(z)$ are LBP with d_n, g_n as recurrence coefficients. Then for arbitrary $q \neq 0$ the polynomials $\tilde{P}_n(z) = q^{-n} P_n(qz)$ are also LBP with the scaled recurrence parameters $\tilde{b}_n = b_n/q, \tilde{d}_n = d_n/q$. The proof of this statement is obvious. This property means that multiplying the recurrence coefficients by the same constant merely leads to a rescaling of the argument z of the polynomials $P_n(z)$. Using this property we can put $\mu = 1$ in formulae (5.11), rewriting them in the form

$$d_n = -\frac{n+\beta}{n+\alpha+1} \qquad g_n = -\frac{n(n+\alpha+\beta)}{(n+\alpha)(n+\alpha+1)} \tag{5.17}$$

where

$$\alpha = \nu + \kappa \qquad \beta = \nu - \kappa - 1.$$

It is interesting to note that LBP with recurrence coefficients (5.17) were considered by Hendriksen and Rossum [14]. These special LBP possess many remarkable properties: they are expressible in terms of the Gauss hypergeometric function [14]

$$P_n(\lambda) = \frac{(\beta)_n}{(1+\alpha)_n}\, {}_2F_1\left(\begin{matrix} -n, \alpha+1 \\ 1-\beta-n \end{matrix}; \lambda\right). \tag{5.18}$$

They are 'classical' in the sense that their derivatives $P_{n+1}'(z)/(n+1)$ are also LPB of the same type with modified parameters α, β [31]. Moreover, these polynomials satisfy the linear second-order differential equation

$$z(1-z)P_n''(z) + (1 - b - n - (2 + a - n)z)P_n'(z) + n(a+1)P_n(z) = 0 \tag{5.19}$$

which belongs to the linear pencil (4.1) with $\mu_n = n$.

The Hendriksen–van Rossum LBP satisfy the biorthogonality property (5.13), where the contour Γ coincides with the unit circle and the weight function is [14]

$$w(z) = (-z)^{-1-\beta}(1-z)^{\alpha+\beta}. \tag{5.20}$$

We see that property (4.3) is crucial for the bispectrality of the corresponding LBP. If this property does not hold (i.e. the four parameters $a_0^{(i)}, a_3^{(i)}, i = 0, 1$ are arbitrary complex numbers) then the resulting LBP do not satisfy a differential equation in the argument λ. Moreover, as is easily seen from (5.16), the dual LBP $P_n^*(\lambda)$ have recurrence coefficients d_n^*, g_n^* which cannot be reduced to the same functional form as d_n, g_n. Hence, for generic choice of parameters $a_0^{(i)}, a_3^{(i)}, i = 0, 1$ the dual polynomials $P_n^*(\lambda)$ do not belong to the sl_2 pencil. We thus have

Proposition 2. *Condition (4.3) for the case of linear pencil (5.6) is equivalent to one of the following statements:*

(i) the resulting LBP are the Hendriksen–van Rossum LBP (5.18);
(ii) the dual LBP $P_n^*(\lambda)$ can also be obtained from the linear pencil (5.6) for the same representation of sl_2;
(iii) the LBP possess the bispectral property, i.e. they satisfy a differential equation of the form (5.19), which in our case is reduced to the form (1.4), where B and C are differential operators of order two and one, respectively.

Assume now that the condition (5.7) does not hold, i.e. all parameters $a_i^{(0,1)}$ are arbitrary (condition (5.4) still holds). Then condition (4.3) is no longer valid, and our LBP are not bispectral, i.e. the polynomials $P_n(\lambda)$ do not satisfy a linear differential equation in λ.

The recurrence coefficients g_n, d_n in the three-term recurrence relation (5.10) are expressed as

$$d_n = \mu_1 \frac{n+\alpha}{n+\beta} \qquad g_n = \mu_2 \frac{n(n+2\nu-1)}{(n+\beta)(n+\beta-1)} \tag{5.21}$$

where now all parameters $\mu_1, \mu_2, \alpha, \beta$ are arbitrary. It is easily seen that the 'dual' recurrence coefficients g_n^*, d_n^* obtained by (5.16) have a more complicated structure than (5.21) and hence the dual polynomials $P_n^*(\lambda)$ cannot in general, be obtained from $P_n(\lambda)$ by a simple change of parameters.

We have thus obtained a *new class of Laurent biorthogonal polynomials* with explicitly known recurrence coefficients (5.21). The polynomials $P_n(\lambda)$ themselves can be expressed in terms of the Gauss hypergeometric functions through (3.13). As far as we know these LBP have not been described in the literature.

Thus we see that the bispectral property (4.3) distinguishes 'classical' LBP from all other 'non-classical' LBP which can be obtained from the linear pencil (5.6). The problem of finding explicit orthogonality property (5.13) for the 'non-classical' LBP with recurrence coefficients (5.21) is still open.

6. Special case: orthogonal polynomials

In this section we consider a special case when the linear operator pencil on the sl_2 algebra gives rise to ordinary orthogonal polynomials. This corresponds to the degenerate case of the previous section when

$$a_2^{(0)} a_1^{(1)} - a_1^{(0)} a_2^{(1)} = 0. \tag{6.1}$$

Then, using linear transformations it is possible to reduce the linear pencil (5.2) to a form in which operator M does not contain 'non-diagonal' parts K_-, K_+, i.e. we can put

$$A_1(\lambda) = a_1 \qquad A_2(\lambda) = a_2 \qquad A_0(\lambda) = a_0^{(0)} - \lambda a_0^{(1)} \qquad A_3(\lambda) = a_3^{(0)} - \lambda a_3^{(1)} \tag{6.2}$$

where a_i are some constants.

For the corresponding coefficients $E_n(\lambda)$ we then have the recurrence relation (3.6)

$$\begin{aligned} a_1(n+1)E_{n+1}(\lambda) &+ \left((n+\nu)a_0^{(0)} + a_3^{(0)}\right)E_n(\lambda) + a_2(n+2\nu-1)E_{n-1}(\lambda) \\ &= \lambda\left(a_0^{(1)}(n+\nu) + a_3^{(1)}\right)E_n(\lambda). \end{aligned} \tag{6.3}$$

To avoid trivialities, we will assume that $a_1 a_2 a_3^{(1)} \neq 0$.

Assume first that $a_0^{(1)} \neq 0$. Then introducing new functions $P_n(\lambda) = \left(a_1/a_0^{(1)}\right)^n n!/\left(\nu + a_3^{(1)}/a_0^{(1)}\right)_n E_n(\lambda)$ we obtain

$$P_{n+1}(\lambda) + u_n P_{n-1} + b_n P_n(\lambda) = \lambda P_n(\lambda) \tag{6.4}$$

with initial condition

$$P_0 = 1 \qquad P_1(\lambda) = \lambda - b_0 \tag{6.5}$$

where

$$\begin{aligned} b_n &= \frac{a_0^{(0)}(n+\nu) + a_3^{(0)}}{a_0^{(1)}(n+\nu) + a_3^{(1)}} \\ u_n &= a_1 a_2 \frac{n(n+2\nu-1)}{\left(a_0^{(1)}(n+\nu) + a_3^{(1)}\right)\left(a_0^{(1)}(n+\nu-1) + a_3^{(1)}\right)}. \end{aligned} \tag{6.6}$$

The recurrence relation (6.4) is a standard three-term recurrence relation which determines orthogonal polynomials. Thus in the special case (6.1) the linear pencil on sl_2 gives rise to some system of orthogonal polynomials.

In the special case $a_0^{(0)} = a_3^{(0)} = 0$ (i.e. when $b_n \equiv 0$) these orthogonal polynomials were introduced and studied in [3]. The authors [3] call them 'random walk' polynomials because of some relations with probability models in mathematical physics (see details in [3]). In [3] explicit orthogonality relation was obtained. For the general case we obtain some new 'non-classical' polynomials with yet unknown orthogonality measure.

If $a_0^{(1)} = 0$ we obtain a standard eigenvalue problem for sl_2

$$(a_1 K_- + a_2 K_+ + a_0 K_0)\psi(\lambda) = \lambda\psi(\lambda) \tag{6.7}$$

since in this case we can put $a_3^{(1)} = 1$ (this leads to rescaling of the remaining coefficients). The corresponding orthogonal polynomials coincide with the Meixner, Pollaczek and Laguerre polynomials (see, e.g., [10, 12]). Note that the Laguerre polynomials correspond to the degenerate case (3.16).

The bispectral property (4.3) for the choice (6.2) is not fulfilled (for $a_1 a_2 \neq 0$). The only exception is the degenerate case corresponding to the Laguerre polynomials.

7. Polynomial pencil on the oscillator algebra

For the oscillator algebra we can choose the representation (2.7) with $c = 0$ (i.e. the case of the Heisenberg–Weyl algebra). The polynomial pencil on the oscillator algebra is defined as

$$(B_1(\lambda)N_- + B_2(\lambda)N_+ + B_0(\lambda)N_0 + B_3(\lambda))\psi(\lambda) = 0 \tag{7.1}$$

where $B_i(\lambda), i = 0, \ldots, 3$ are polynomials in λ. Expanding $\psi = \sum_{n=0}^{\infty} C_n(\lambda)e_n$, we arrive at the recurrence relation

$$\sqrt{n+1}B_1(\lambda)C_{n+1}(\lambda) + B_2\sqrt{n}C_{n-1}(\lambda) + (B_0(\lambda)n + B_3(\lambda))C_n(\lambda) = 0. \tag{7.2}$$

Omitting details, we present here the final expression for $C_n(\lambda)$. If $B_0(\lambda) \neq 0$ then

$$C_n(\lambda) = \frac{(-B_2/B_0)^n}{\sqrt{n!}} {}_2F_0\left(\begin{matrix} -n, \frac{B_3B_0-B_1B_2}{B_0^2} \\ - \end{matrix}; -B_0^2\big/(B_1B_2)\right). \tag{7.3}$$

Obviously the resulting functions $C_n(\lambda)$ will be bispectral if condition

$$B_1B_2 - B_3B_0 = \kappa B_0^2 \tag{7.4}$$

holds where κ is a constant not depending on λ. Indeed, if the condition (7.4) is valid then the parameters of the hypergeometric function in (7.3) do not depend on λ and hence there is a differential equation of second order for the coefficients $C_n(\lambda)$.

If $B_0(\lambda) = 0$ then we have

$$C_n(\lambda) = \left(\sqrt{\frac{2B_2y^2}{B_1n!}}\right)^n {}_2F_0\left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix}; -1/y^2\right) \tag{7.5}$$

where $y = -B_3/\sqrt{2B_1B_2}$. This case will always be bispectral.

Consider a linear pencil on the oscillator algebra, i.e. $B_i(\lambda) = b_i^{(0)} - \lambda b_i^{(1)}, i = 0, \ldots, 3$ are linear in λ. As in the case of the sl_2 algebra, we should distinguish two cases: (i) $b_2^{(0)}b_1^{(1)} - b_1^{(0)}b_2^{(1)} \neq 0$ and (ii) $b_2^{(0)}b_1^{(1)} - b_1^{(0)}b_2^{(1)} = 0$. In the first case, using the same technique as in the previous section, we can reduce the problem to the Laurent biorthogonal polynomials (5.10) with the recurrence coefficients

$$d_n = -\frac{\alpha n + \beta}{\gamma n + \delta} \qquad g_n = -\frac{n}{(\gamma n + \delta)(\gamma(n-1) + \delta)} \tag{7.6}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters. The bispectral property (7.4) leads to two possibilities: either $\gamma = 0, \delta = 1/\alpha$, or $\alpha = 0, \beta = 1/\gamma$. Using scaling transformations for LBP we can reduce the first possibility to the case

$$d_n = -n - \beta \qquad g_n = -n \tag{7.7}$$

and the second possibility to the case

$$d_n = -\frac{1}{n+\beta} \qquad g_n = -\frac{n}{(n+\beta)(n+\beta-1)}. \tag{7.8}$$

It is interesting to note that the first possibility (7.7) corresponds to the so-called Appell LBP [28]. This means that the only monic LBP $P_n(z)$ with the Appell property $P_n'(z) = nP_{n-1}(z)$ are LBP with recurrence coefficients (7.7).

Explicitly the Appell LBP are given by the expression

$$P_n(z) = (\beta)_n {}_1F_1\left(\begin{matrix} -n, \\ 1-\beta-n \end{matrix}; z\right). \tag{7.9}$$

They satisfy differential equation

$$zP_n''(z) + (1-\beta-z)P_n'(z) = n(P_n'(z) - P_n(z)) \tag{7.10}$$

which has the form of differential linear pencil (with $\mu_n = n$ as spectral parameter). Thus, the Appell LBP satisfy the differential linear equation in the variable z and difference linear equation in the variable n.

The second possibility (7.8) corresponds to the LBP which are dual biorthogonal partners with respect to the Appell LBP given by (7.7). This is verified directly using formulae (5.16). We thus have

Proposition 3. *For the case of Laurent biorthogonal polynomials associated with the oscillator algebra, the bispectrality condition (7.4) characterizes (up to scaling transformation) either Appell LBP or their dual biorthogonal partners.*

Consider now the degenerate case $b_2^{(0)} b_1^{(1)} - b_1^{(0)} b_2^{(1)} = 0$. Then it is possible to reduce the linear pencil on the oscillator algebra to the recurrence relation for orthogonal polynomials (6.4) with recurrence coefficients

$$b_n = \frac{\xi n + \eta}{\alpha n + \beta} \qquad u_n = \frac{\gamma n}{(\alpha n + \beta)(\alpha(n-1) + \beta)} \tag{7.11}$$

where $\alpha, \beta, \gamma, \xi, \eta$ are arbitrary parameters. If $\xi = \eta = 0$, i.e. $b_n = 0$ then we obtain polynomials which are known in the literature as the Tricomi–Carlitz orthogonal polynomials [3]. These polynomials are remarkable as 'non-classical' OP (i.e. they do not satisfy any differential or finite-difference equation with respect to the argument z) having explicitly known expression and orthogonality relation (for details see, e.g., [3]). As far as we know, a natural origin of the Tricomi–Carlitz polynomials in frames of the GEVP for the oscillator algebra was not yet observed previously.

For generic case (7.11) the orthogonality measure for corresponding polynomials is still unknown. Finally, if $\alpha = 0$, we obtain Charlier (if $\xi \neq 0$) or Hermite (if $\xi = 0$) polynomials. Only the latter ones possess the bispectral property.

8. Application: integrable chains

Generalized eigenvalue problems—in contrast to the ordinary eigenvalue problems—are not yet so popular in dealing with problems in mathematical and theoretical physics. In this section we give an example connected with the so-called relativistic Toda chain and its generalization.

The relativistic Toda chain was proposed by Ruijsenaars in [20]. It can be written out as a system of two evolution equations for two unknowns b_n, g_n:

$$\dot{g}_n = \frac{b_n}{g_{n-1}} - \frac{b_{n+1}}{g_{n+1}} \qquad \dot{b}_n = b_n(1/g_{n-1} - 1/g_n). \tag{8.1}$$

The restricted RTC is characterized by the condition $b_0 = 0, n \geqslant 0$. It was shown in [16] that equations (8.1) can be obtained as a compatibility condition LBP $P_n(z;t)$ satisfying the recurrence relation (5.10) and the evolution equation

$$\dot{P}_n(z;t) = b_n/g_n P_{n-1}(z;t). \tag{8.2}$$

On the other hand, the equations for RTC can be obtained from the Lax pair ansatz (see, e.g. [26, 27]) as follows. Assume that we have a linear pencil (5.3) for a pair of operators L, M. If the operators $L(t), M(t)$ and eigenfunction $\psi(t)$ depend on time t in such a manner that for some operators G_1, G_2 the condition

$$\dot{L} = LG_1 - G_2 L \qquad \dot{M} = MG_1 - G_2 M \tag{8.3}$$

and $\dot{\psi}(\lambda) = -G_1\psi$ are satisfied, then the pencil (5.10) is *isospectral*, i.e. $\dot{\lambda} = 0$. Concrete realizations of this Lax pair ansatz generate different integrable systems connected with LBP and their generalization—polynomials of R_I—type [27].

We consider here a simple concrete example. Let $L = \alpha_1 N_- + \alpha_0 N_0 + \alpha_3$, $M = \beta_2 N_+ + \beta_0 N_0 + \beta_3$, where $N_0, N_\pm$ are generators of the oscillator algebra and the parameters α_i, β_i depend on t. Choose $G_1 = G_2 = N_+$. Then it is easily verified that the solution of the ansatz (8.3) is $L = \alpha(N_- + t - t_0)$, $M = \beta(N_0 + (t - t_1)N_+ + \kappa)$, where $\alpha, \beta, \kappa, t_0, t_1$ are arbitrary parameters not depending on t. Putting $t_1 = t_0 = 0$ and $\alpha = \beta$ we see that the linear pencil (5.3) is equivalent to the recurrence relation for LBP (5.10) with recurrence coefficients

$$g_n = \frac{t}{\kappa + n} \qquad b_n = \frac{tn}{(\kappa + n)(\kappa + n - 1)}. \tag{8.4}$$

Comparing (8.4) with (7.8) we see that corresponding polynomials $P_n(z;t)$ are the scaled Appel LBP

$$\frac{t^n}{(\kappa)_n} {}_2F_0\left(\begin{matrix}-n, \kappa\\ -\end{matrix}; tz\right). \tag{8.5}$$

It is interesting to note that the recurrence coefficients $d_n(t), g_n(t)$ admit separation of variables t and n:

$$d_n = \epsilon_1(t)\delta_n \qquad g_n = \epsilon_2(t)\gamma_n \tag{8.6}$$

where δ_n, γ_n do not depend on t whereas $\epsilon_1(t), \epsilon_2(t)$ do not depend on n. In [16] it was shown that all such solutions for the RTC are reduced to the form (8.4).

Thus we have shown that solutions of the restricted RTC with separated variables correspond to a linear isospectral pencil on the oscillator algebra.

9. Conclusion

We have found that a linear pencil on either sl_2 or the oscillator algebra gives rise to families of orthogonal polynomials and Laurent biorthogonal polynomials. Apart from getting some well-known results (e.g. the OP of the Sheffer class are obtained from the ordinary eigenvalue problem for sl_2) we have obtained new results:

(i) the classical Hendriksen–van Rossum LBP appear from the linear pencil on sl_2 with an additional bispectral condition;
(ii) the Appell LBP appear from the linear pencil on the oscillator algebra with an additional bispectral condition;
(iii) solutions for relativistic Toda chain with separated variables appear from the Lax pair for a linear pencil on the oscillator algebra.

Moreover we found some new systems of LBP arising from the linear pencil on sl_2 or the oscillator algebra which do not possess the bispectral property. These polynomials have explicit recurrence coefficients and can be expressed in terms of hypergeometric function ${}_2F_1$, but their orthogonality measure is yet unknown.

We restricted ourselves only to the bispectral problem when the operators B, C in (1.4) are differential ones. It would be interesting to consider a more complicated case when B, C are difference operators (e.g. tri-diagonal matrices).

Another possible generalization of the problem can be obtained by considering more complicated algebras (say $su(3)$ or the so-called 'quantum algebras'). We are planning to consider this problem separately.

Note that a generating function similar to (3.7) was considered in [19], where the authors considered only the case of orthogonal polynomials. We have shown here that this generating function appears naturally from the study of a polynomial pencil on sl_2 and that moreover, there exists interesting classes of Laurent biorthogonal polynomials arising from this pencil.

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